

Attempt the questions yourself before looking at the hints or solutions!

Question:

6 Show that

$$\int_0^1 \frac{x^4}{1+x^2} dx = \frac{\pi}{4} - \frac{2}{3}.$$

Determine the values of

(i) $\int_0^1 x^3 \tan^{-1} \left(\frac{1-x}{1+x} \right) dx,$

(ii) $\int_0^1 \frac{(1-y)^3}{(1+y)^5} \tan^{-1} y dy.$

Worked Solution: (Want a video solution? [Click here](#))

STEP Maths 2 2001

$$\begin{aligned}
 (6) \quad & \int_0^1 \frac{x^4}{1+x^2} dx \\
 &= \int_0^1 \frac{x^4 + x^2 - x^2}{1+x^2} dx \\
 &= \int_0^1 \left\{ x^2 - \frac{x^2 + 1 - 1}{1+x^2} \right\} dx \\
 &= \int_0^1 \left\{ x^2 - 1 + \frac{1}{1+x^2} \right\} dx \\
 &= \left[\frac{x^3}{3} - x + \tan^{-1} x \right]_0^1 \\
 &= \left(\frac{1}{3} - 1 + \frac{\pi}{4} \right) \\
 &= \frac{\pi}{4} - \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 (i) \quad & \int_0^1 x^3 \tan^{-1} \frac{1-x}{1+x} dx \\
 &= \left[\frac{x^4}{4} \tan^{-1} \frac{1-x}{1+x} \right]_0^1 \\
 &\quad - \int_0^1 \frac{x^4}{4} \cdot \frac{1}{1 + \left(\frac{1-x}{1+x} \right)^2} \cdot \frac{-(1+x) - (1-x)}{(1+x)^2} dx \\
 &= \int_0^1 \frac{x^4}{2} \cdot \frac{1}{(1+x)^2 + (1-x)^2} dx \\
 &= \int_0^1 \frac{x^4}{2} \cdot \frac{1}{2 + 2x^2} dx \\
 &= \frac{1}{4} \int_0^1 \frac{x^4}{1+x^2} dx \\
 &= \frac{\pi}{16} - \frac{1}{6}
 \end{aligned}$$

$$(ii) I = \int_0^1 \frac{(1-y)^3}{(1+y)^5} \tan^{-1} y dy$$

$$= \frac{1}{4} x^3 (1+x)^2$$

$$\begin{aligned}
 \tan^{-1} y \\
 &= \tan^{-1} \frac{1-x}{1+x}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{-(1+x) - (1-x)}{(1+x)^2} \\
 &= \frac{-2}{(1+x)^2}
 \end{aligned}$$

so I

$$\begin{aligned}
 &= \int_1^0 \frac{1}{4} x^3 (1+x)^2 \cdot \tan^{-1} \frac{1-x}{1+x} \cdot \frac{-2}{(1+x)^2} dx \\
 &= \frac{1}{2} \int_0^1 x^3 \tan^{-1} \left(\frac{1-x}{1+x} \right) dx \\
 &= \frac{1}{2} \left(\frac{\pi}{16} - \frac{1}{6} \right) \\
 &= \frac{\pi}{32} - \frac{1}{12}
 \end{aligned}$$

Put

$$y = \frac{1-2x}{1+2x}; \text{ then}$$

$$\frac{(1-y)^3}{(1+y)^5}$$

$$= \frac{\left(1 - \frac{1-2x}{1+2x}\right)^3}{\left(1 + \frac{1-2x}{1+2x}\right)^5}$$

$$= \frac{(2x)^3}{2^5} \cdot (1+2x)^2$$

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