Attempt the questions yourself before looking at the hints or solutions!

Question:

6 Show that

$$\int_0^1 \frac{x^4}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4} - \frac{2}{3}\,.$$

Determine the values of

(i) $\displaystyle\int_0^1 x^3\ \tan^{-1}\left(\frac{1-x}{1+x}\right)\ \mathrm{d}x\,,$

(ii) $\displaystyle\int_0^1 \frac{(1-y)^3}{(1+y)^5}\tan^{-1}y\ \mathrm{d}y\,.$

Worked Solution: (Want a video solution? Click here)

STEP Maths 2 2001

(6) $\int_0^1 \frac{x^4}{1+x^2}\,dx$

$= \int_0^1 \frac{x^4+x^2-x^2}{1+x^2}\,dx$

$= \int_0^1 \left\{ x^2 - \frac{x^2+1-1}{1+x^2} \right\} dx$

$= \int_0^1 \left\{ x^2 - 1 + \frac{1}{1+x^2} \right\} dx$

$= \left[\frac{x^3}{3} - x + \tan^{-1} x \right]_0^1$

$= \left(\frac{1}{3} - 1 + \frac{\pi}{4} \right)$

$= \frac{\pi}{4} - \frac{2}{3}$

(i) $\int_0^1 x^3 \tan^{-1} \frac{1-x}{1+x}\,dx$

$= \left[\frac{x^4}{4} \tan^{-1} \frac{1-x}{1+x} \right]_0^1$

$- \int_0^1 \frac{x^4}{4} \cdot \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \frac{-(1+x)-(1-x)}{(1+x)^2}\,dx$

$= \int_0^1 \frac{x^4}{2} \cdot \frac{1}{(1+x)^2+(1-x)^2}\,dx$

$= \int_0^1 \frac{x^4}{2} \cdot \frac{1}{2+2x^2}\,dx$

$= \frac{1}{4} \int_0^1 \frac{x^4}{1+x^2}\,dx$

$= \frac{\pi}{16} - \frac{1}{6}$

(ii) $I = \int_0^1 \frac{(1-y)^3}{(1+y)^5} \tan^{-1} y\,dy$

$= \frac{1}{4} x^3 (1+x)^2$

$\tan^{-1} y$

$= \tan^{-1} \frac{1-x}{1+x}$

and

$\frac{dy}{dx} = \frac{-(1+x)-(1-x)}{(1+x)^2}$

$= \frac{-2}{(1+x)^2}$

so I

$= \int_1^0 \frac{1}{4} x^3 (1+x)^2 \cdot \tan^{-1} \frac{1-x}{1+x} \cdot \frac{-2}{(1+x)^2}\,dx$

$= \frac{1}{2} \int_0^1 x^3 \tan^{-1}\left(\frac{1-x}{1+x}\right) dx$

$= \frac{1}{2} \left(\frac{\pi}{16} - \frac{1}{6} \right)$

$= \frac{\pi}{32} - \frac{1}{12}$

Put

$y = \dfrac{1-x}{1+x}$; then

$$\frac{(1-y)^3}{(1+y)^5}$$

$$= \frac{\left(1-\frac{1-x}{1+x}\right)^3}{\left(1+\frac{1-x}{1+x}\right)^5}$$

$$= \frac{(2x)^3}{2^5}\cdot(1+x)^2$$