Attempt the questions yourself before looking at the hints or solutions!

Question:

6 Two curves are given parametrically by

(1) $$x_1 = (\theta + \sin\theta), \quad y_1 = (1 + \cos\theta),$$

and

(2) $$x_2 = (\theta - \sin\theta), \quad y_2 = -(1 + \cos\theta).$$

Find the gradients of the tangents to the curves at the points where $\theta = \pi/2$ and $\theta = 3\pi/2$.

Sketch, using the same axes, the curves for $0 \leqslant \theta \leqslant 2\pi$.

Find the equation of the normal to the curve (1) at the point with parameter θ. Show that this normal is a tangent to the curve (2).

Worked Solution: (Want a video solution? Click here)

98 STEP 9

(b) $x_1 = \theta + \sin\theta$

$\Rightarrow \dfrac{dx_1}{d\theta} = 1 + \cos\theta$

$y_1 = 1 + \cos\theta$

$\Rightarrow \dfrac{dy_1}{d\theta} = -\sin\theta$

so $\dfrac{dy_1}{dx_1} = \dfrac{-\sin\theta}{1+\cos\theta}$

$= \dfrac{2\cos\frac{1}{2}\theta\sin\frac{1}{2}\theta}{-2\cos^2\frac{1}{2}\theta}$

$= -\tan\frac{1}{2}\theta$

$x_2 = \theta - \sin\theta$

$\Rightarrow \dfrac{dx_2}{d\theta} = 1 - \cos\theta$

$y_2 = -(1+\cos\theta)$

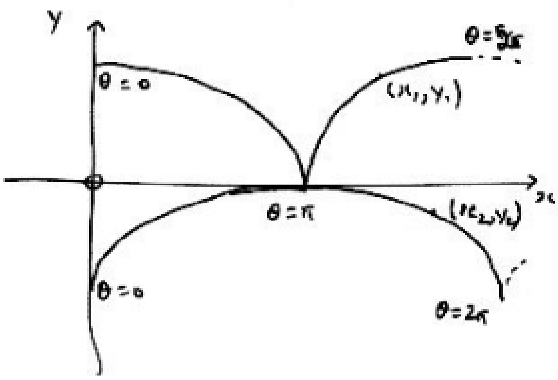
$\Rightarrow \dfrac{dy_2}{d\theta} = \sin\theta$

so $\dfrac{dy_2}{dx_2} = \dfrac{\sin\theta}{1-\cos\theta}$

$= \dfrac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}$

$= \cot\frac{1}{2}\theta$

θ	$\pi/2$	$3\pi/2$
$\dfrac{dy_1}{dx_1}$	-1	$+1$
$\dfrac{dy_2}{dx_2}$	1	-1



Normal to curve 1 at point with parameter θ is

$$y - 1 - \cos\theta = \cot\tfrac{1}{2}\theta\,(x - \theta - \sin\theta)$$

Now substitute

$x = \theta - \sin\theta$

$\Rightarrow y - 1 - \cos\theta = -\cot\frac{1}{2}\theta \cdot 2\sin\theta$

$\Rightarrow y - 2\cos^2\frac{1}{2}\theta = -\dfrac{\cos\frac{1}{2}\theta \cdot 4\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$

$= -4\cos^2\frac{1}{2}\theta$

$\Rightarrow y = -2\cos^2\frac{1}{2}\theta$

$= -(1+\cos\theta)$

Hence $(\theta - \sin\theta, -[1+\cos\theta])$ (a point on the second curve) lies on the normal to the first at $(\theta + \sin\theta, 1+\cos\theta)$. Moreover the gradient of this line is $\cot\frac{1}{2}\theta$, also the gradient of the tangent to the second curve at $(\theta - \sin\theta, -[1+\cos\theta])$. Thus the normal to the first is also a tangent to the second. QED.