Attempt the questions yourself before looking at the hints or solutions!

Question:

4 The integral I_n is defined by

$$I_n = \int_0^{\pi} (\pi/2 - x)\sin(nx + x/2)\,\mathrm{cosec}\,(x/2)\,\mathrm{d}x,$$

where n is a positive integer. Evaluate $I_n - I_{n-1}$, and hence evaluate I_n leaving your answer in the form of a sum.

Worked Solution: (Want a video solution? Click here)

(4) $I_n = \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \sin\left(nx + \frac{x}{2}\right) \operatorname{cosec} \frac{x}{2}\, dx$

$I_{n-1} = \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \sin\left(nx - \frac{x}{2}\right) \operatorname{cosec} \frac{x}{2}\, dx$

$\Rightarrow I_n - I_{n-1}$

$= \int_0^{\pi} 2\left(\frac{\pi}{2} - x\right) \cos nx \sin \frac{x}{2} \operatorname{cosec} \frac{x}{2}\, dx$

$= \int_0^{\pi} 2\left(\frac{\pi}{2} - x\right) \cos nx\, dx$

$= \left[\frac{2}{n}\left(\frac{\pi}{2} - x\right) \sin nx\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi} \sin nx\, dx$

$= \frac{2}{n^2}\left[-\cos nx\right]_0^{\pi}$

$= \frac{2}{n^2}\left[-(-1)^n + 1\right]$

$\Rightarrow I_{2n+1} - I_{2n}$

$= \frac{4}{(2n+1)^2}$ $\qquad I_{2n} - I_{2n-1} = 0$

$\Rightarrow I_{2n+1} - I_{2n-1}$

$= \frac{4}{(2n+1)^2} \quad (n \geq 1)$

$\Rightarrow I_{2n+1} = \sum_{r=1}^{n} \frac{4}{(2n+1)^2} + I_1$

$I_{2n} = I_{2n-1} \quad (n \geq 1)$

$= \sum_{r=1}^{n-1} \frac{4}{(2n+1)^2} + I_1$

$I_1 = 4 + I_0$

$= \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \sin \frac{x}{2} \operatorname{cosec} \frac{x}{2}\, dx + 4$

$= \left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi} + 4$

$= 4$

Hence

$I_{2n+1} = \sum_{r=0}^{n} \frac{4}{(2r+1)^2}$; $I_{2n} = \sum_{r=0}^{n-1} \frac{4}{(2r+1)^2}$