Attempt the questions yourself before looking at the hints or solutions!

Question:

13 Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.

(ii) Find the probability that exactly two husbands sit next to their wives.

(iii) Find the probability that no husband sits next to his wife.

Examiners' Report:

Q13 This combinatorics question was attempted by close to half of all candidates, a very encouraging statistic. About two-thirds of the attempts did not progress very far, gaining five marks or fewer, but of those who did get further, the marks were fairly evenly distributed.

For part (i), most attempts reached the stated answer, although a significant number used very creative, if inaccurate or meaningless, ways of doing so. The majority of candidates used counting methods, and many of these were successful to a greater or lesser extent. The other method used by many candidates was to consider the probability of the first wife sitting next to her husband ($\frac{2}{5}$) and the conditional probability of the spouse of the other person sitting next to the first husband sitting next to them (this is $\frac{1}{3}$), and then multiplying these.

It is crucial at this point to reinforce that candidates must explain their reasoning in their answers, especially when they are working towards a given answer. Simply writing $\frac{2}{5}\times 13=\frac{2}{15}$ is woefully inadequate to gain all of the available marks; there must be a justification of the reasoning behind it.

Parts (ii) and (iii) were found to be a lot more challenging. A number of candidates attempted to construct probabilistic arguments, which are very challenging in this case. The successful attempts all used pure counting arguments. The examiners often found it challenging to decipher their thinking, though, as the explanations were often somewhat incoherent. Those who used counting arguments usually made good progress on both parts.

The favoured method for part (iii) was to use P(no pairs) = 1 − P(≥ 1 pair). It would have certainly been worth checking the answer obtained using a direct method, as this would have caught a number of errors.

The main errors encountered in good attempts at the later parts of the question were a failure to consider all possible cases or a miscounting of the number of ways each possible case could occur.

Overall, this question was answered well by a significant number of candidates.

Official Solution: (Want a video solution? Click here)

Question 13

Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.

We call the couples H_1 and W_1, H_2 and W_2, H_3 and W_3. We seat H_1 arbitrarily, leaving $5! = 120$ ways of seating the remaining five people. If each husband sits next to his wife, then there are two seats in which W_1 can sit, each of which leaves four consecutive seats for the other two couples.

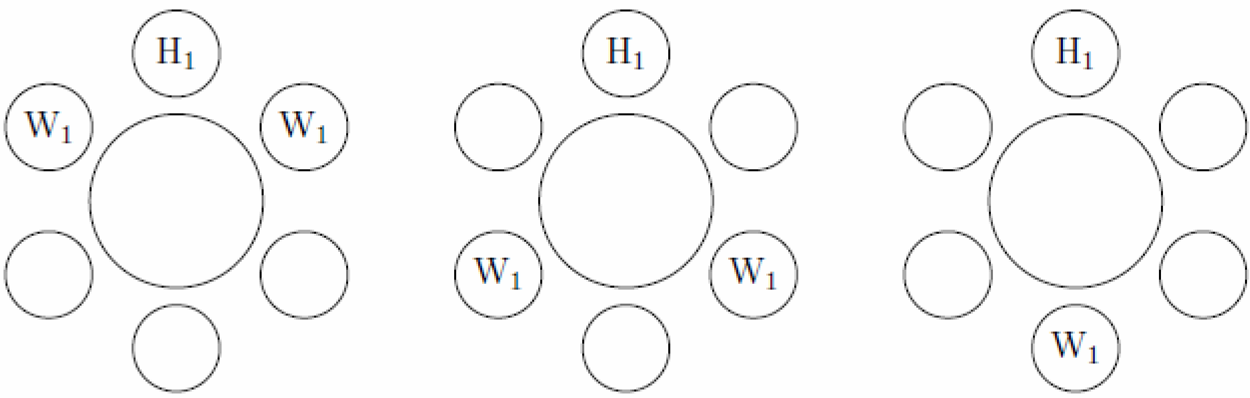
There are four choices for who to sit immediately next to W_1, and that forces the following seat as well (being the spouse of that person).

Next, there are two choices for who to seat the other side of H_1, and the final person must sit next to their spouse.

So there are $2 \times 4 \times 2 = 16$ ways to have all of the husbands sitting next to their wives, with a probability of $\frac{16}{120} = \frac{2}{15}$.

(ii) Find the probability that exactly two husbands sit next to their wives.

Assume to begin with that H_1 and W_1 are separated and the other two husbands are seated next to their wives. Once H_1 has been seated, there are five possible positions for W_1, as shown in the following diagrams (there are two possibilities shown in each of the first two):



Clearly the first two possibilities do not work: in the first, H_1 and W_1 are next to each other. In the second, whoever sits between H_1 and W_1 will be separated from their spouse. So W_1 must sit opposite H_1, one couple sits to the right of H_1 and the other couple to his left. There are two choices for which couple sits to the right of H_1, and two choices for whether the husband or wife sits next to H_1; similarly there are two choices for whether the husband or wife of the third couple sits next to H_1. So in all, there are $2 \times 2 \times 2 = 8$ ways to seat the couples with H_1 and W_1 separated and the other couples together.

Similarly, there are 8 ways with couple 2 separated and 8 ways with couple 3 separated, so there are $3 \times 8 = 24$ ways in total.

Thus the probability is $\frac{24}{120} = \frac{3}{15} = \frac{1}{5}$.

(iii) *Find the probability that no husband sits next to his wife.*

Method 1: First find the probability of exactly one husband sitting next to his wife.

Let us assume that H_3 and W_3 are the only pair next to each other. Then in the above diagrams, the left hand one fails as H_1 and W_1 are together. The right hand one also fails, as if H_3 and W_3 are together, H_2 and W_2 must also be. So the only valid configuration is the middle one, with either H_2 or W_2 between H_1 and W_1 and the other partner on the other side of either H_1 or W_1.

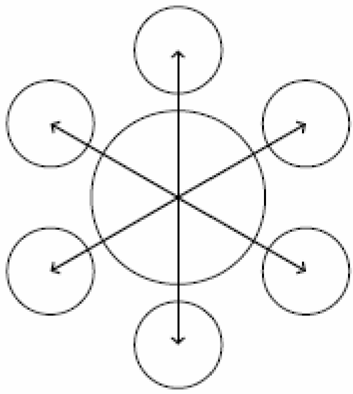
There are two choices for where W_1 will sit, two choices for which of H_2 or W_2 will sit between H_1 and W_1, two choices for where the other partner will sit, and two choices for which way round H_3 and W_3 will sit, giving $2 \times 2 \times 2 \times 2 = 16$ possibilities.

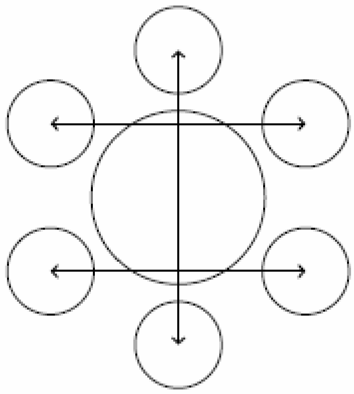
Finally, we need to multiply this by three, as any one of the three couples could be the adjacent one, giving $3 \times 16 = 48$ possibilities, and hence a probability of $\frac{48}{120} = \frac{2}{5} = \frac{6}{15}$.

Thus the probability that no husband sits next to his wife is $1 - \frac{2}{15} - \frac{3}{15} - \frac{6}{15} = \frac{4}{15}$.

Method 2: Find the probability directly.

If no husband sits next to his wife, there are two possible configurations as shown in these diagrams (the arrows join husbands with their wives):





The number of ways of arranging the first case (fixing H_1 at the top as usual) is $2 \times 2 \times 2 = 8$, as their are two ways of choosing which couple sits in which of the diagonal pairs of seats, two ways of couple 2 sitting and two ways of couple 3 sitting.

For the second case, which is no longer totally symmetrical between the three couples, if H_1 sits in the top seat, there are again 8 ways of seating the other two couples. As there are three choices for which couple sits opposite each other, there are $3 \times 8 = 24$ ways in all.

Thus in total there are $8 + 24 = 32$ ways, giving a total probability of $\frac{32}{120} = \frac{4}{15}$.

Worked Solution: (Want a video solution? Click here)

(13) STEP Maths 1 08S

Throughout we do not distinguish rotations of the same configuration. On that basis there are $5!$ $= 120$ arrangements without constraint since the first person may sit anywhere;

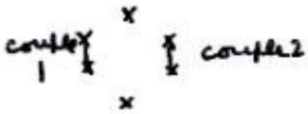
(i) One husband can sit down anywhere. Then his wife can sit down in either of two places. Then there is a choice of two couples to their left. That couple can sit in one of two ways. Then the remaining couple may sit in two ways.

Hence probability

$$= \frac{2\times2\times2\times2}{5!}$$

$$= \frac{2}{15}$$

(ii) The configuration has to be of the form



There are three choices for the couples which sit together. One of the husbands of those may sit anywhere. Then there are two options for his wife.

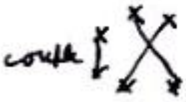
The other paired couple may sit down opposite in two ways, the unpaired couple may sit in the remaining seats in two ways.

Probability

$$= \frac{3 \times 2 \times 2 \times 2}{120}$$

$$= \frac{1}{5}$$

(iii) We seek first the probability of exactly one couple sitting together. There are three choices of couple, and they may sit anywhere in two ways.



Then this is the only configuration.

There are two options for the couples, and each couple has two ways of sitting. So probability

$$= \frac{3 \times 2 \times 2 \times 2 \times 2}{120}$$

$$= \frac{2}{5}$$

Hence the no pair option has probability

$$1 - \frac{2}{15} - \frac{1}{5} - \frac{2}{5}$$

$$= \frac{15 - 2 - 3 - 6}{15}$$

$$= \frac{4}{15}$$

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