Attempt the questions yourself before looking at the hints or solutions!

Question:

11 Two long circular cylinders of equal radius lie in equilibrium on an inclined plane, in contact with one another and with their axes horizontal. The weights of the upper and lower cylinders are W_1 and W_2, respectively, where $W_1 > W_2$. The coefficients of friction between the inclined plane and the upper and lower cylinders are μ_1 and μ_2, respectively, and the coefficient of friction between the two cylinders is μ. The angle of inclination of the plane is α (which is positive).

(i) Let F be the magnitude of the frictional force between the two cylinders, and let F_1 and F_2 be the magnitudes of the frictional forces between the upper cylinder and the plane, and the lower cylinder and the plane, respectively. Show that $F = F_1 = F_2$.

(ii) Show that

$$\mu \geqslant \frac{W_1 + W_2}{W_1 - W_2},$$

and that

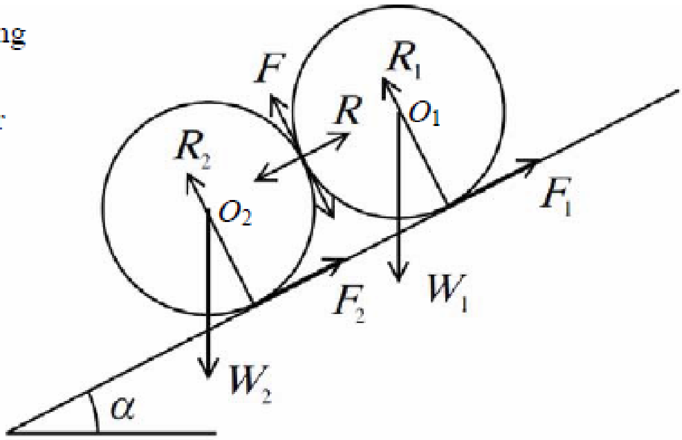
$$\tan\alpha \leqslant \frac{2\mu_1 W_1}{(1+\mu_1)(W_1+W_2)}.$$

Examiners' Report:

Q11 attracted double the number of attempts of Q10, but had a marginally lower mean score, and this was slightly surprising. A few years ago, statics questions such as this would have been gobbled up with glee by many candidates, happy to collect some very easy mechanics marks. The great hurdle for the weaker candidates remains the widespread inability to draw a good diagram with all relevant forces marked on it in appropriate directions. Sadly, here, almost all diagrams failed to include all of the relevant forces, and decent progress beyond that point was, therefore, essentially impossible. Remarkably few candidates managed even to explain satisfactorily that the two frictional forces were equal (by taking moments about the central axis of each of the two cylinders).

Official Solution: (Want a video solution? Click here)

Q11 As with all statics problems, the key to getting a good start, and to making life as easy as possible for the working that follows, is to have a good, clear diagram with all relevant forces, in appropriate directions, marked on it (see alongside).



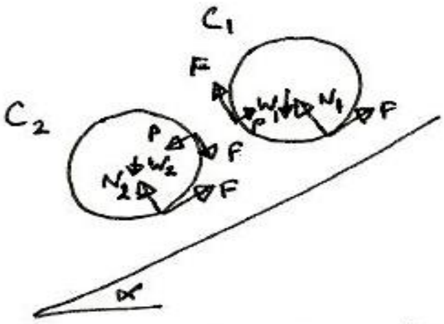
To begin with, take moments about the respective cylinders' axes yields $F = F_1 = F_2$, as required. Next, write down the four equations that arise from resolving for each cylinder in the directions parallel and perpendicular to the plane.

These are:-

$F_1 + R = W_1 \sin\alpha$ ①; $R_1 + F = W_1 \cos\alpha$ ②; $F_2 - R = W_2 \sin\alpha$ ③ and $R_2 - F = W_2 \cos\alpha$ ④.

(Note that one could replace some of these with equivalent equations gained from resolving for the whole system.) Replacing F_1 and F_2 by F, equating for $\sin\alpha$, re-arranging for F in terms of R and using the Friction Law, $F \le \mu R$, appropriately leads to the first given answer in (ii). A bit more determination is needed to gain the second given answer, however. Firstly, $\tan\alpha$ can be gained by division in at least two ways, and both F and R must be eliminated from any equations being used. Thereafter, it is simply a matter of forcing the working through correctly and, hopefully, concisely.

Worked Solution: (Want a video solution? Click here)



By moments about centres respectively, all friction forces are equal.

(i) R $\parallel$ slope for $C_1 \Rightarrow$

$$P + F = W_1 \sin\alpha$$

R $\parallel$ slope for $C_2 \Rightarrow$

$$F = W_2 \sin\alpha + P$$

$$\Rightarrow P = (W_1 - W_2)\sin\alpha - P$$

$$\Rightarrow 2P = (W_1 - W_2)\sin\alpha$$

$$\Rightarrow P = \tfrac{1}{2}(W_1 - W_2)\sin\alpha$$

From same equations

$$P + 2F = (W_1 + W_2)\sin\alpha + P$$

$$\Rightarrow F = \tfrac{1}{2}(W_1 + W_2)\sin\alpha$$

$$\text{So } \frac{F}{P} = \frac{W_1 + W_2}{W_1 - W_2}$$

$$\Rightarrow \mu \geqslant \frac{W_1 + W_2}{W_1 - W_2} \quad \text{QED.}$$

(ii) R $\perp$ plane for $C_1 \Rightarrow$

$$F + N_1 = W_1 \cos\alpha$$

$$\Rightarrow N_1 = W_1 \cos\alpha - F$$

$$\Rightarrow \frac{F}{N_1} = \frac{F}{W_1\cos\alpha - F}$$

$$= \frac{\tfrac{1}{2}(W_1 + W_2)\sin\alpha}{W_1\cos\alpha - \tfrac{1}{2}(W_1 + W_2)\sin\alpha}$$

$$= \frac{W_1 + W_2}{2W_1\cot\alpha - W_1 - W_2}$$

$$\Rightarrow \mu_1 \geqslant \frac{W_1 + W_2}{2W_1\cot\alpha - W_1 - W_2}$$

$$\Rightarrow 2\mu_1 W_1 \cot\alpha - \mu_1(W_1 + W_2) \geqslant W_1 + W_2$$

(assuming $N_1 \geqslant 0$)

$$\Rightarrow 2\mu_1 W_1 \cot\alpha \geqslant (\mu_1 + 1)(W_1 + W_2)$$

$$\Rightarrow 2\mu_1 W_1 \geqslant (1 + \mu_1)(W_1 + W_2)\tan\alpha$$

$$\Rightarrow \tan\alpha \leqslant \frac{2\mu_1 W_1}{(1 + \mu_1)(W_1 + W_2)} \quad \text{QED}$$