Attempt the questions yourself before looking at the hints or solutions!

Question:

3 For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points P_1, P_2, ... are defined by $P_1 = X * Y$ and, for $n \geqslant 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

Examiners' Report:

Q3 This vector question was actually very straightforward, though its unfamiliar appearance clearly put most candidates off, with only around 350 of them making an attempt at it. There were nine marks available for the first two parts, which were technically undemanding, and it is no coincidence that the mean score on the question was around 9½/20. I suspect that, for the most part, this was considered by candidates to be one of those questions that are done towards the end of the examination in order to bump up their paper total by getting the easier marks at the beginning of the question, with no real intention of making a complete attempt. Candidates usually gave up part-way through (iii) where a stab at the "corresponding result for $X * (Y * Z)$" was required of them, which was actually just $(X * Y) * (X * Z)$. I imagine this highlights the lack of students' familiarity with such properties as distributivity when considering binary operations.

Official Solution: (Want a video solution? Click here)

Q3 This vector question is tied up with the geometric understanding that, for distinct points with position vectors $\mathbf{x}$ and $\mathbf{y}$, the point with p.v. $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ cuts XY in the ratio $(1-\lambda):\lambda$ (though it is important to realise that this point is only between X and Y if $0 < \lambda < 1$). Part (i) tests (algebraically) the property of *commutativity* (whether the composition yields different results if the order of the application of the operation is changed):

$$X*Y = Y*X \quad \Leftrightarrow \lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \Leftrightarrow (2\lambda - 1)(\mathbf{x}-\mathbf{y}) = \mathbf{0} \Leftrightarrow (\text{since } \mathbf{x} \neq \mathbf{y}) \ \ \lambda = \tfrac{1}{2}.$$

Part (ii) then explores the property of *associativity* (whether the outcome is changed when the order of the elements involved in two successive operations remains the same but the pairings within those successive operations is different). Here we have

$$(X*Y)*Z = \lambda\,(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + (1-\lambda)\mathbf{z} = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$$

and $$X*(Y*Z) = \lambda\mathbf{x} + (1-\lambda)[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}] = \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$$

Thus, $(X*Y)*Z - X*(Y*Z) = \lambda(1-\lambda)(\mathbf{x}-\mathbf{z})$ and the two are distinct provided $\lambda \neq 0, 1$ or $X \neq Z$.

Part (iii) now explores a version of the property of *distributivity* (although usually referring to two distinct operations): $(X*Y)*Z = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$, and

$$(X*Z)*(Y*Z) = [\lambda\mathbf{x} + (1-\lambda)\mathbf{z}]*[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}] = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{z} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$$
$$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}, \text{ and the two are always equal.}$$

Next, $X*(Y*Z) = \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$, and

$$\begin{aligned}(X*Y)*(X*Z) &= [\lambda\mathbf{x} + (1-\lambda)\mathbf{y}]*[\lambda\mathbf{x} + (1-\lambda)\mathbf{z}] \\ &= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + \lambda(1-\lambda)\mathbf{x} + (1-\lambda)^2\mathbf{z} \\ &= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}.\end{aligned}$$

Hence $X*(Y*Z) = (X*Y)*(X*Z)$ also.

In (iv), you will notice that the condition $0 < \lambda < 1$ comes into play, so that P_1 cuts XY *internally* in the ratio $(1-\lambda):\lambda$. Following this process through a couple more steps shows us that P_n cuts XY in the ratio $(1-\lambda^n):\lambda^n$, which is easily established inductively.

Worked Solution: (Want a video solution? Click here)

(i) Let $U \equiv X*Y$, $V \equiv Y*X$

Then $\mathbf{u} = \lambda \mathbf{x} + (1-\lambda)\mathbf{y}$

$\mathbf{v} = \lambda \mathbf{y} + (1-\lambda)\mathbf{x}$

So $\mathbf{u} = \mathbf{v}$

$\Rightarrow \mathbf{0} = (1-2\lambda)\mathbf{x} + (2\lambda-1)\mathbf{y}$

$= (1-2\lambda)(\mathbf{x}-\mathbf{y})$

But $\mathbf{x} \neq \mathbf{y}$, so we have

$\lambda = \frac{1}{2}$

Hence if $\lambda \neq \frac{1}{2}$, we cannot have $\mathbf{u} = \mathbf{v}$, so $X*Y$ and $Y*X$ are distinct.

(ii) Let $P = (X*Y)*Z$

$Q = X*(Y*Z)$

Then

$\mathbf{p} = \lambda(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + (1-\lambda)\mathbf{z}$

$= \lambda^2\mathbf{x} + (1-\lambda)\lambda\mathbf{y} + (1-\lambda)\mathbf{z}$

and $\mathbf{q} = \lambda\mathbf{x} + (1-\lambda)(\lambda\mathbf{y} + (1-\lambda)\mathbf{z})$

$= \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$

So $\mathbf{p} = \mathbf{q}$

$\Rightarrow \mathbf{0} = (\lambda^2-\lambda)\mathbf{x} + \lambda(1-\lambda)\mathbf{z}$

$= \lambda(1-\lambda)(\mathbf{x}-\mathbf{z})$

$\Rightarrow \lambda = 0$ or $\lambda = 1$ or $\mathbf{x} = \mathbf{z}$

Then are distinct if $\lambda \neq 0$ and $\lambda \neq 1$ and $\mathbf{x} \neq \mathbf{z}$

(iii) Let $H = (X*Y)*Z$

$K = (X*Z)*(Y*Z)$

then $\mathbf{h} = \lambda(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + (1-\lambda)\mathbf{z}$

$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$

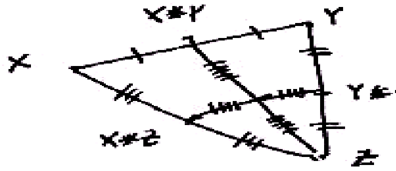
and $\mathbf{k} = \lambda[\lambda\mathbf{x} + (1-\lambda)\mathbf{z}] + (1-\lambda)[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}]$

$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{z} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$

$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$

$\Rightarrow \mathbf{h} = \mathbf{k}$ i.e. $H \equiv K$ QED.

$\lambda = \frac{1}{2}$ gives the mid-point.



Let $H = X*(Y*Z)$

and $K = (X*Y)*(X*Z)$

then $\mathbf{h} = \lambda\mathbf{x} + (1-\lambda)(\lambda\mathbf{y} + (1-\lambda)\mathbf{z})$

$= \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$

$\mathbf{k} = \lambda[\lambda\mathbf{x} + (1-\lambda)\mathbf{y}] + (1-\lambda)[\lambda\mathbf{x} + (1-\lambda)\mathbf{z}]$

$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + \lambda(1-\lambda)\mathbf{x} + (1-\lambda)^2\mathbf{z}$

$= \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$

$\Rightarrow \mathbf{h} = \mathbf{k}$ ie $H = K$

So $X*(Y*Z) = (X*Y)*(X*Z)$

$\mathbf{p}_1 = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$

$\mathbf{p}_2 = \lambda[\lambda\mathbf{x} + (1-\lambda)\mathbf{y}] + (1-\lambda)\mathbf{y}$

$= \lambda^2\mathbf{x} + (1+\lambda)(1-\lambda)\mathbf{y}$

$\mathbf{p}_3 = \lambda[\lambda^2\mathbf{x} + (1+\lambda)(1-\lambda)\mathbf{y}] + (1-\lambda)\mathbf{y}$

$= \lambda^3\mathbf{x} + (1+\lambda+\lambda^2)(1-\lambda)\mathbf{y}$

Thus $\mathbf{p}_n = \lambda^n\mathbf{x} + (1+\lambda+\dots+\lambda^{n-1})(1-\lambda)\mathbf{y}$

$= \lambda^n\mathbf{x} + (1-\lambda^n)\mathbf{y}$

$\Rightarrow \overrightarrow{XP_n} = \lambda^n\mathbf{x} + (1-\lambda^n)\mathbf{y} - \mathbf{x}$

$= (1-\lambda^n)(\mathbf{y}-\mathbf{x})$

and $\overrightarrow{P_nY}$

$= \mathbf{y} - \lambda^n\mathbf{x} - (1-\lambda^n)\mathbf{y}$

$= \lambda^n(\mathbf{y}-\mathbf{x})$

so $XP_n : P_nY$

$= (1-\lambda^n) : \lambda^n$
