Attempt the questions yourself before looking at the hints or solutions!

Question:

13 In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

(ii) Let d be the value of X such that $\mathrm{P}(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.

(iii) Given that $d = \sqrt{2}$, find a.

Examiners' Report:

Question 13

This question was attempted by around 20% of candidates, but was only counted as one of the best six questions for about three-quarters of them. Even among those, it was the most poorly answered question on the whole paper, with one-third of the attempts gaining only 0 or 1 mark, a median of 4 marks and an upper quartile of 8 marks.

STEP I 2011 Principal Examiner's Report

(i) In this first part, many candidates did not even attempt to find k, and algebraic errors during the integration were common. A number of attempts to find k were obviously wrong, as they gave a negative area, but candidates did not notice this.

Most were unable to integrate $x\sqrt{4-x^2}$, with failed attempts to integrate by parts far outnumbering correct integrations of this expression.

A number of candidates attempted to calculate the median rather than the mean. Other bizarre interpretations of the term "mean" were also seen.

(ii) Few candidates made it this far. Of those who did, there were some very good solutions. One of the hardest parts was getting the logic correct: the phrasing of the question as "Show that X if Y" was often misinterpreted to mean "Show that if X then Y", though the majority of the marks were awarded in such cases.

(iii) The handful of students who made it this far were generally successful at this part, too.

Official Solution: (Want a video solution? Click here)

Question 13

In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

We know that $\int_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x = 1$, so we begin by performing this integration to determine k. We have

$$\begin{aligned}\int_{-a}^{0} 2k\,\mathrm{d}x + \int_{0}^{2} k\sqrt{4-x^2}\,\mathrm{d}x &= \big[2kx\big]_{-a}^{0} + k\big[2\arcsin\tfrac{x}{2} + \tfrac{1}{2}x\sqrt{4-x^2}\,\big]_0^2 \\ &= 2ak + k\big((2\arcsin 1 + 0) - (2\arcsin 0 + 0)\big) \\ &= 2ak + k\pi \\ &= k(2a+\pi) \\ &= 1,\end{aligned}$$

so $k = 1/(2a+\pi)$.

We can now work out the mean of X; we work in terms of k until the very end to avoid ugly calculations. We can integrate the expression $x\sqrt{4-x^2} = kx(4-x^2)^{\frac{1}{2}}$ either using inspection (as we do in the following) or the substitution $u = 4-x^2$, giving $\mathrm{d}u/\mathrm{d}x = -2x$, so that the integral becomes $k\int_4^0 -\frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u = k\big[-\frac{1}{3}u^{\frac{3}{2}}\big]_4^0 = \frac{8}{3}k$.

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\mathrm{f}(x)\,\mathrm{d}x \\
&= \int_{-a}^{0} 2kx\,\mathrm{d}x + \int_{0}^{2} kx\sqrt{4-x^2}\,\mathrm{d}x \\
&= \left[kx^2\right]_{-a}^{0} + \left[-\tfrac{k}{3}(4-x^2)^{\frac{3}{2}}\right]_{0}^{2} \\
&= (0-ka^2) + (0-(-\tfrac{k}{3}\times 4^{\frac{3}{2}})) \\
&= -ka^2 + \tfrac{8}{3}k \\
&= \frac{\frac{8}{3}-a^2}{2a+\pi} \\
&= \frac{8-3a^2}{3(2a+\pi)}.
\end{aligned}$$

STEP I 2011 Question 13 continued

> *(ii)* *Let d be the value of X such that $\mathrm{P}(X>d)=\frac{1}{10}$. Show that $d<0$ if $2a>9\pi$ and find an expression for d in terms of a in this case.*

We have $d<0$ if and only if $\mathrm{P}(X>0)<\mathrm{P}(X>d)=\frac{1}{10}$, so we consider $\mathrm{P}(X>0)$. Using the above integration (or noting that X is uniform for $x<0$), we have

$$\begin{aligned}
\mathrm{P}(X>0) &= 1-\mathrm{P}(X<0) \\
&= 1-2ak \\
&= 1-\frac{2a}{2a+\pi} \\
&= \frac{\pi}{2a+\pi}.
\end{aligned}$$

Therefore $\mathrm{P}(X>0)<\frac{1}{10}$ if and only if

$$\frac{\pi}{2a+\pi}<\frac{1}{10};$$

that is $10\pi<2a+\pi$, or $2a>9\pi$. Putting these together gives $d<0$ if and only if $2a>9\pi$.

In this case, as $d<0$, we have

$$\mathrm{P}(X>d)=1-\mathrm{P}(X<d)=1-2k(d-(-a)),$$

so $1-2k(d+a)=\frac{1}{10}$, so $d+a=\frac{9}{10}/2k$, giving

$$\begin{aligned}
d &= \frac{9}{20k}-a \\
&= \frac{9(2a+\pi)}{20}-a \\
&= \frac{9\pi-2a}{20}.
\end{aligned}$$

Note that, since $2a > 9\pi$, this gives us $d < 0$ as we expect.

An alternative approach is to calculate the cumulative distribution function first. We have

$$\mathrm{F}(x) = \begin{cases} 0 & x < -a \\ 2k(x+a) & -a \leqslant x < 0 \\ k\big(2a + 2\arcsin \frac{1}{2}x + \frac{1}{2}x\sqrt{4-x^2}\big) & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

(though only the part with $-a \leqslant x \leqslant 0$ is actually needed).

Then we solve $\mathrm{F}(d) = \frac{9}{10}$. If it turns out that $d < 0$, then we have $\mathrm{F}(d) = 2k(d+a) = \frac{9}{10}$, which rearranges to give $d = (9\pi - 2a)/20$ as above. Now if $2a > 9\pi$, then $(9\pi - 2a)/20 < 0$ so that $F((9\pi - 2a)/20) = \frac{9}{10}$ and $d < 0$, as required.

STEP I 2011 Question 13 continued

(iii) Given that $d = \sqrt{2}$, find a.

We note that now $d > 0$, so we have to integrate to find a explicitly. We get

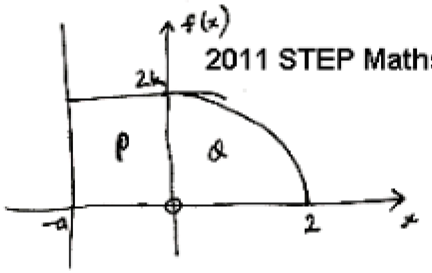
$$\begin{aligned} \mathrm{P}(X > \sqrt{2}) = \int_{\sqrt{2}}^{2} k\sqrt{4-x^2}\,\mathrm{d}x &= \big[2\arcsin \tfrac{x}{2} + \tfrac{1}{2}x\sqrt{4-x^2}\,\big]_{\sqrt{2}}^{2} \\ &= k\big((2\arcsin 1 + 0) - (2\arcsin \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\sqrt{4-2})\big) \\ &= k(\pi - \tfrac{\pi}{2} - 1) \\ &= k(\tfrac{\pi}{2} - 1) \\ &= \frac{\frac{\pi}{2} - 1}{2a + \pi} \\ &= \tfrac{1}{10}. \end{aligned}$$

Thus $10(\frac{\pi}{2} - 1) = 2a + \pi$, so that $2a = 4\pi - 10$, giving our desired result: $a = 2\pi - 5$.

Alternatively, one could calculate $\mathrm{P}(X < \sqrt{2})$ in the same manner and find a such that this equals $\frac{9}{10}$.

As a check, it is clear that $2a = 4\pi - 10 < 9\pi$, so $d \geqslant 0$ from part (ii).

Worked Solution: (Want a video solution? Click here)



2011 STEP Maths 1 no 13

Total area is 1, so

$$1 = 2ka + k\int_0^2 \sqrt{(4-x^2)}\,dx$$

$$= 2ka + k\left[2\arcsin\frac{x}{2} + \frac{1}{2}x\sqrt{(4-x^2)}\right]_0^2$$

$$= 2ka + k\pi$$

$$\Rightarrow k = \frac{1}{2a+\pi}$$

(i) $$E[X] = \int_{-a}^{0} 2kx\,dx + \int_0^2 kx\sqrt{(4-x^2)}\,dx$$

$$= \left[kx^2\right]_{-a}^{0} + \left[-\frac{1}{3}k(4-x^2)^{3/2}\right]_0^2$$

$$= -ka^2 + \frac{8k}{3}$$

$$= \frac{(-3a^2+8)}{3(2a+\pi)}$$

$$= \frac{8-3a^2}{3(2a+\pi)}$$

(ii) $d < 0$ iff area $P > 0.9$.

i.e $0.9 < \frac{2a}{2a+\pi}$ (from earlier result)

$\Leftrightarrow 18a + 9\pi < 20a$

$\Leftrightarrow 9\pi < 2a$

i.e. $d < 0$ if $2a > 9\pi$.

In that case we have

$2k(a+d) = 0.9$

$\Rightarrow a + d = \frac{0.9(2a+\pi)}{2}$

$\Rightarrow d = \frac{9a}{10} + \frac{9\pi}{20} - a$

$= \frac{9\pi}{20} - \frac{a}{10}$

$= \frac{9\pi - 2a}{20}.$

(iii) Now take $d = \sqrt{2}$. So

$$0.9 = \int_{-a}^{0} 2k \, dx + \int_{0}^{\sqrt{2}} k\sqrt{(4-x^2)}\,dx$$

$$= 2ka + k\left[2\arcsin\frac{x}{2} + \frac{1}{2}x\sqrt{(4-x^2)}\right]_0^{\sqrt{2}}$$

$$= 2ka + k\left[\frac{\pi}{2} + \frac{1}{2}\sqrt{2}\sqrt{2}\right]$$

$$= 2ka + k\left[\frac{\pi}{2} + 1\right]$$

$$= \frac{2a}{2a+\pi} + \frac{\frac{\pi}{2}+1}{2a+\pi}$$

$\Rightarrow 18a + 9\pi = 20a + 5\pi + 10$

$\Rightarrow 2a = 4\pi - 10$

$\Rightarrow a = 2\pi - 5$