Attempt the questions yourself before looking at the hints or solutions!

Question:

1 What does it mean to say that a number x is *irrational*?

Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.

B: If $p+q$ is irrational, then at least one of p and q is irrational.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p+q$ is irrational.

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi+\mathrm{e}$, $\pi-\mathrm{e}$, $\pi^2-\mathrm{e}^2$, $\pi^2+\mathrm{e}^2$ is rational.

Examiners' Report:

Q1 This question was primarily about logical thinking and structuring an argument. While it was a very popular question, the marks were disappointing: only 30% of candidates gained more than 6 marks.

Most candidates could describe vaguely what is meant by the term irrational, though only a handful gave a precise, accurate definition. The popular offering of 'a number with an infinite decimal expansion' was not acceptable.

It was pleasing to see, though, that the majority of candidates were capable of using proof by contradiction to prove statements A and B, and they then went on to provide a counterexample to statement C. A small number of very strong candidates justified their counterexamples by proving that the numbers they presented were in fact irrational, though any well-known irrational examples were given full marks without the need for justification.

It is important to stress the difference between proving a statement and disproving one; while a single (numerical) counterexample is adequate to disprove a statement, a proof of the truth of a statement requires a general argument. Too many candidates wrote things such as: 'If $pq = \sqrt{3}$, then $p = \sqrt{3}$ and $q = 1$, so . . . .' Also, it is unknown what an irrational number 'looks like', so the frequently occurring arguments such as 'We know that $e + \pi$ must be irrational because the numbers are not of the same form' (when comparing this example to something like $(1 - \sqrt{2}) + \sqrt{2}$) are spurious.

Sadly, very few candidates made any significant progress on the main part of the question. Several attempted (unsuccessfully) to prove that all four of the given numbers are irrational. Others asserted that since π and e are both irrational, π + e must also be despite having just disproved statement C. A number of candidates successfully showed that π + e and π - e cannot both be irrational by appealing to B, but then could not see how to continue. The best attempts proceeded by using A and B repeatedly to show that no pair of $\pi \pm e$ and $\pi^2 \pm e^2$ could simultaneously be rational (that is, they considered all six cases separately).

Official Solution: (Want a video solution? Click here)

Question 1

What does it mean to say that a number x is irrational?

It means that we cannot write $x = m/n$ where m and n are integers with $n \neq 0$.

Prove by contradiction statements A and B below, where p and q are real numbers.

***A:** If pq is irrational, then at least one of p and q is irrational.*

***B:** If $p+q$ is irrational, then at least one of p and q is irrational.*

We first prove statement A.

Assume that pq is irrational, but neither p nor q is irrational, so that both p and q are rational. But then pq is the product of two rational numbers, so is rational. This contradicts that assumption that pq is irrational. So statement A is true.

Now for statement B we argue similarly.

Assume that $p+q$ is irrational, but neither p nor q is irrational, so that both p and q are rational. But then $p+q$ is the sum of two rational numbers, so is rational. This contradicts the assumption that $p+q$ is irrational. So statement B is true.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

***C:** If p and q are irrational, then $p+q$ is irrational.*

One example is $p = \sqrt{2}$, $q = -\sqrt{2}$.

If the numbers e, π, π^2, e^2 and $e\pi$ are irrational, prove that at most one of the numbers $\pi + e$, $\pi - e$, $\pi^2 - e^2$, $\pi^2 + e^2$ is rational.

We assume that the five given numbers are, indeed, irrational.

We have $(\pi + e) + (\pi - e) = 2\pi$, which is irrational (if p is irrational, then so is $2p$). So by statement B, at least one of $\pi + e$ and $\pi - e$ is irrational.

Similarly, $(\pi^2 + e^2) + (\pi^2 - e^2) = 2\pi^2$, which is irrational. So by statement B again, at least one of $\pi^2 + e^2$ and $\pi^2 - e^2$ is irrational.

Assume that both $\pi + e$ and $\pi^2 - e^2$ are rational. Then

Worked Solution: (Want a video solution? Click here)

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A rational number r can be expressed as a ratio of two integers p, q, i.e.

$$r = \frac{p}{q} \quad (q \neq 0) \quad p, q \in \mathbb{Z}.$$

An irrational number cannot by definition be expressed in that way.

Proof by contradiction of proposition A.

Suppose ab is irrational, and both a and b are rational.

Then $a = \frac{p_a}{q_a}$, $b = \frac{p_b}{q_b}$ as above, where $p_a, q_a, p_b, q_b \in \mathbb{Z}$ (and $q_a \neq 0$, $q_b \neq 0$). As a result

$$ab = \frac{p_a p_b}{q_a q_b}$$

where $p_a p_b$, $q_a q_b (\neq 0)$ are integers. That makes ab rational — a contradiction. Hence at least one of p, q has to be irrational.

Proof by contradiction of proposition B.

Suppose $a+b$ is irrational and also a and b are rational.

Then (as before)

$$a+b = \frac{p_a}{q_a} + \frac{p_b}{q_b} = \frac{p_a q_b + q_a p_b}{q_a q_b}$$

where p_a, etc, are integers.

Then $p_a q_b + q_a p_b$, $q_a q_b$ are integers, so $a+b$ is rational — again a contradiction. Hence at least one of p, q is rational.

Disproof by counterexample of proposition C.

$p = \sqrt{2}$, $q = -\sqrt{2}$ (both irrational) but

$p + q = 0 = \frac{0}{1}$ so is rational. That is a counter-example.

$(\pi+e)^2 = \pi^2+2\pi e+e^2$
$(\pi-e)^2 = \pi^2-2\pi e+e^2$
$\pi^2-e^2 = (\pi+e)(\pi-e)$
$\pi^2+e^2 = (\pi+e)^2-2\pi e$

Consider the following cases of supposed rational pairs.

a) $\pi+e$, $\pi-e$ are rational
Then $\pi+e+\pi-e$
$= 2\pi$ is rational — a contradiction.

b) $\pi+e$, π^2-e^2 are rational
then $\dfrac{\pi^2-e^2}{\pi+e}$
$= \pi-e$ is rational – case a) again

c) $\pi+e$, π^2+e^2 are rational
π^2+e^2
$= (\pi+e)^2-2\pi e$
$\Rightarrow$ $2\pi e = (\pi+e)^2-\pi^2-e^2$ is rational - contradiction.

d) $\pi-e$, π^2-e^2 are rational
then $\dfrac{\pi^2-e^2}{\pi-e}$
$= \pi+e$ is rational — case a) again.

e) $\pi-e$, π^2+e^2 are rational.
$2\pi e = (\pi^2+e^2)-(\pi-e)^2$ is rational - contradiction

f) π^2-e^2, π^2+e^2 are rational, then
$\pi^2+e^2+\pi^2-e^2$
$= 2\pi^2$ is rational — a contradiction.

That exhausts all possible cases.