Attempt the questions yourself before looking at the hints or solutions!

Question:

6 **(i)** Given that $x^2 - y^2 = (x-y)^3$ and that $x - y = d$ (where $d \neq 0$), express each of x and y in terms of d. Hence find a pair of integers m and n satisfying $m - n = (\sqrt{m} - \sqrt{n})^3$ where $m > n > 100$.

(ii) Given that $x^3 - y^3 = (x-y)^4$ and that $x - y = d$ (where $d \neq 0$), show that $3xy = d^3 - d^2$. Hence show that

$$2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

and determine a pair of distinct positive integers m and n such that $m^3 - n^3 = (m-n)^4$.

Examiners' Report:

6 This was a popular, straightforward question, which was often answered well. However, algebraic errors still occurred, for example when expanding $(x-y)^3$.

Official Solution: (Want a video solution? Click here)

6 (i) Since $x^2 - y^2 \equiv (x-y)(x+y)$, the given equation reduces to $d(x+y) = d^3$.

Hence $x + y = d^2$, and also we are given that $x - y = d$.

Therefore $x = \frac{1}{2}(d^2 + d)$ and $y = \frac{1}{2}(d^2 - d)$.

For the second equation, let $x = \sqrt{m}$ and $y = \sqrt{n}$ and let $d = 6$ (any choice of $d \geqslant 6$ will work).

Then $x = 21$ and $y = 15$, so $m = 441$ and $n = 225$.

You might like to check: $441 - 225 = 216 = (21 - 15)^3$.

(ii) It helps here to know that $x^3 - y^3 \equiv (x-y)(x^2 + xy + y^2)$.

Therefore the equation $x^3 - y^3 = (x-y)^4$ reduces to $x^2 + xy + y^2 = d^3$.

We know that $x^2 - 2xy + y^2 = d^2$ since $x - y = d$.

Hence, subtracting these two, $3xy = d^3 - d^2$.

This result can also be deduced by simplifying $x^3 - (x-d)^3 = d^4$ (using $x - y = d$).

Since $d = x - y$

$\Rightarrow 3x(x-d) = d^3 - d^2$

$\Rightarrow 3x^2 - 3dx - (d^3 - d^2) = 0$

$$\Rightarrow x = \frac{3d \pm \sqrt{9d^2 + 12(d^3 - d^2)}}{6} = \frac{3d \pm d\sqrt{12d - 3}}{6}$$

$$\Rightarrow 2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

For x to be integer we need $\frac{4d-1}{3}$ to be a perfect square.

If $d = 1$ then either $x = 0$ (not permitted) or $x = 1$ which implies $y = 0$ (not permitted).

So let $d = 7$ (for example), since $4 \times 7 - 1 = 27 = 3 \times 3^2$ ($d = 17$ also works).

$\Rightarrow 2x = 7 \pm 7\sqrt{9}$

Therefore $x = 14$ and hence $y = x - d = 7$ (choosing the positive values).

You might like to check: $14^3 - 7^3 = (2 \times 7)^3 - 7^3 = 7 \times 7^3 = 7^4 = (14 - 7)^4$.

Worked Solution: (Want a video solution? Click here)

(i)

$$x^2 - y^2 = (x-y)^3$$

$$\Rightarrow (x-y)(x+y) = (x-y)(x-y)^2$$

$$\Rightarrow x+y = (x-y)^2 \quad (x \neq y)$$

$$\Rightarrow \begin{cases} x+y = d^2 \\ x-y = d \end{cases}$$

$$\Rightarrow \begin{cases} x = \tfrac{1}{2}(d+d^2) \\ y = \tfrac{1}{2}(d^2-d) \end{cases}$$

Suppose $m - n = (\sqrt{m} - \sqrt{n})^3$ for integers m, n. Let

$$x^2 = m \quad (x > 0)$$

$$y^2 = n. \quad (y > 0)$$

then $x^2 - y^2 = (x-y)^3$

$$\Rightarrow \begin{cases} x = \tfrac{1}{2}(d+d^2) \\ y = \tfrac{1}{2}(d^2-d) \end{cases}$$

We look for a solution where

$$x > y > 10,$$

by trying $d = 1, 2,$ etc. $d = 6$ gives

$$x = 21$$

$$y = 15$$

$$\Rightarrow \begin{cases} m = 441 \\ n = 225 \end{cases}$$

(ii)

$$x^3 - y^3 = (x-y)^4$$

$$\Rightarrow (x-y)(x^2+xy+y^2) = (x-y)(x-y)^3$$

$$\Rightarrow x^2+xy+y^2 = (x-y)^3 \quad (x \neq y)$$

$$\Rightarrow \begin{cases} x^2+xy+y^2 = d^3 \\ x-y = d \end{cases}$$

$$\Rightarrow d^3 - d^2 = x^2+xy+y^2 - (x^2-2xy+y^2)$$

$$= 3xy \quad \text{QED}$$

i.e. $\begin{cases} 3xy = d^3-d^2 \\ x-y = d \end{cases}$

$$\Rightarrow 3x(x-d) = d^3 - d^2$$

$$\Rightarrow 0 = 3x^2 - 3dx - d^3 + d^2$$

$$\Rightarrow x = \frac{3d \pm \sqrt{(9d^2 + 12[d^3-d^2])}}{6}$$

$$= \frac{3d \pm \sqrt{(12d^3 - 3d^2)}}{6d}$$

$$= \frac{d \pm \sqrt{\left(\frac{12d-3}{9}\right)}.d}{2}$$

$$\Rightarrow 2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

QED

Thus we also have

$$2y = -d \pm d\sqrt{\frac{4d-1}{3}}$$

Now consider

$$m^3 - n^3 = (m-n)^4$$

Identify m and n with x and y, respectively, above. Then try successively $d = 1, 2, \ldots$

$d = 7$ gives

$$2x = 7 \pm 7\sqrt{9}$$

$$2y = -7 \pm 7\sqrt{9}$$

Take positive signs; then

$$x = 14,\ y = 7$$

These check!

$$m = 14,\ n = 7$$
