Attempt the questions yourself before looking at the hints or solutions!

Question:

6 (i) Show that, if $(a\,, b)$ is any point on the curve $x^2 - 2y^2 = 1$, then $(3a + 4b\,, 2a + 3b)$ also lies on the curve.

(ii) Determine the smallest positive integers M and N such that, if $(a\,, b)$ is any point on the curve $Mx^2 - Ny^2 = 1$, then $(5a + 6b\,, 4a + 5b)$ also lies on the curve.

(iii) Given that the point $(a\,, b)$ lies on the curve $x^2 - 3y^2 = 1$, find positive integers P, Q, R and S such that the point $(Pa + Qb\,, Ra + Sb)$ also lies on the curve.

Official Solution: (Want a video solution? Click here)

6 (i) The assertion that "$(a\,,\,b)$ lies on the curve $x^2 - 2y^2 = 1$" is equivalent to stating that $a^2 - 2b^2 = 1$. Since $(3a+4b)^2 - 2(2a+3b)^2 \equiv 9a^2 + 24ab + 16b^2 - 2\left(4a^2 + 12ab + 9b^2\right) \equiv a^2 - 2b^2$, the point $(3a+4b\,,\,2a+3b)$ lies on the curve $x^2 - 2y^2 = 1$ if $(a\,,\,b)$ does.

(ii) We are told that $Ma^2 - Nb^2 = 1$, and we want to find M and N so that $M(5a+6b)^2 - N(4a+5b)^2 = 1 = Ma^2 - Nb^2$.

Since $M(5a+6b)^2 - N(4a+5b)^2 \equiv a^2(25M - 16N) + ab(60M - 40N) + b^2(36M - 25N)$, we could let $25M - 16N = M$, $60M - 40N = 0$ and $36M - 25N = -N$.

These are all equivalent to $3M = 2N$, so let $M = 2$ and $N = 3$.

(iii) We require $(Pa + Qb)^2 - 3(Ra + Sb)^2 \equiv a^2 - 3b^2 = 1$.

$\Rightarrow P^2 - 3R^2 = 1$

and $2PQ = 6RS$

and $Q^2 - 3S^2 = -3$

The first of these equations suggests letting $P = 2$ and $R = 1$. Then the second equation reduces to $4Q = 6S$ which suggests letting $Q = 3$ and $S = 2$. These values are consistent with the third equation.

Therefore $(2a + 3b\,,\,a + 2b)$ is the simplest solution.

$(7a + 12b\,,\,4a + 7b)$ is another solution, since $7^2 - 3 \times 4^2 = 1, 7 \times 12 = 3 \times 7 \times 4$, and $12^2 - 3 \times 7^2 = -3$.

Equations of the form $x^2 - dy^2 = 1$ are called Pell equations; the techniques for solving them (which underlie this question) are explained in most undergraduate textbooks on Number Theory. It might be interesting to consider the equation $x^2 - 4y^2 = 1$: what happens when an argument similar to (iii) is pursued?

Worked Solution: (Want a video solution? Click here)

STEP Mathematics 1 2006

(6) (i) If (a,b) lies on the curve, then $1 = a^2 - 2b^2$.

Consider now

$$(3a+4b)^2 - 2(2a+3b)^2 - 1$$

$$= 9a^2 + 24ab + 16b^2 - 2(4a^2 + 12ab + 9b^2) - 1$$

$$= 9a^2 - 8a^2 + 24ab - 24ab + 16b^2 - 18b^2 - 1$$

$$= a^2 - 2b^2 - 1$$

$$= 0 \quad \text{from above.}$$

So $(3a+4b, 2a+3b)$ does indeed lie on the same curve.

(ii) If (a,b) lies on the curve

$$0 = Ma^2 - Nb^2 - 1 \qquad (*)$$

Consider now

$$Z \equiv M(5a+6b)^2 - N(4a+5b)^2 - 1$$

$$= M(25a^2 + 60ab + 36b^2) - N(16a^2 + 40ab + 25b^2) - 1$$

$$= (25M - 16N)a^2 + (60M - 40N)ab + (36M - 25N)b^2 - 1$$

$$= 25(Ma^2 - Nb^2) - 1 + (60M - 40N)ab + 36Mb^2 - 16Na^2$$

$$= 24 - 16Na^2 + 36Mb^2 + 20(3M - 2N)ab$$

C/f with (*) above: if both are identities they must be equivalent.

Hence

$$0 = 3M - 2N$$

$$\frac{16N}{M} = 24$$

$$\frac{36M}{N} = 24$$

all are equivalent to

$$\frac{M}{N} = \frac{2}{3}$$

for which the least positive integral solution is $M=2, N=3$

(iii) $1 = a^2 - 3b^2$

and $1 = (Pa+Qb)^2 - 3(Ra+Sb)^2$

$= P^2a^2 + 2PQab + Q^2b^2 - 3R^2a^2 - 6RSab - 3S^2b^2$

$= (P^2-3R^2)a^2 + 2(PQ-3RS)ab + (Q^2-3S^2)b^2$

Again for two equivalent identities we must have

$1 = P^2 - 3R^2$

$-3 = Q^2 - 3S^2$

$0 = PQ - 3RS$

Only three equations for four unknowns, so try

$P = 2, R = 1$

$\Rightarrow \begin{cases} -3 = Q^2 - 3S^2 \\ 0 = 2Q - 3S \end{cases}$

$\Rightarrow -3 = \frac{9S^2}{4} - 3S^2$

$\Rightarrow -12 = -3S^2$

$\Rightarrow S = 2$ (positive integer required)

$\Rightarrow Q = 3$

Thus a possible set is

$P = 2, Q = 3, R = 1, S = 2$