Attempt the questions yourself before looking at the hints or solutions!

Question:

3 For any number x, the largest integer less than or equal to x is denoted by $[x]$. For example, $[3.7] = 3$ and $[4] = 4$.

Sketch the graph of $y = [x]$ for $0 \leqslant x < 5$ and evaluate

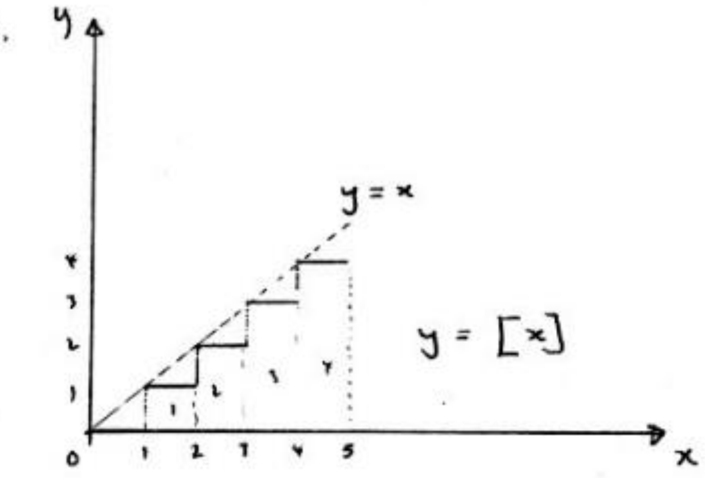
$$\int_0^5 [x] \, dx.$$

Sketch the graph of $y = [e^x]$ for $0 \leqslant x < \ln n$, where n is an integer, and show that

$$\int_0^{\ln n} [e^x] \, dx = n \ln n - \ln(n!).$$

Worked Solution: (Want a video solution? Click here)

3.



$y = [x]$

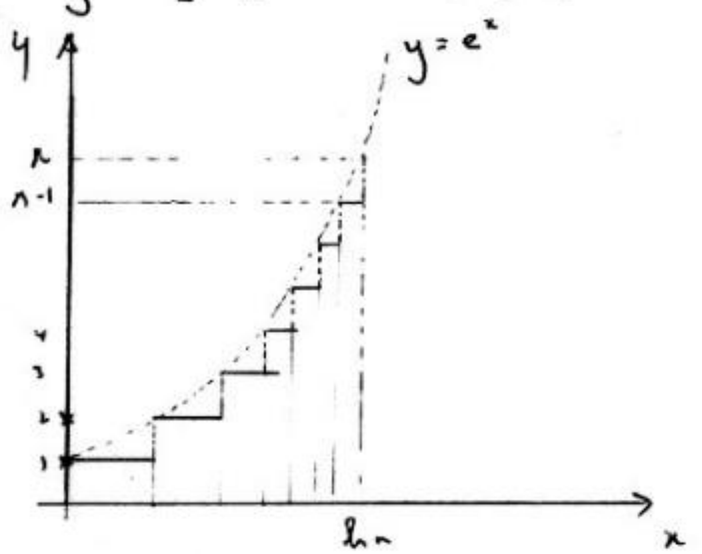
$0 \le x < 5$

$$\int_0^5 [x] \, dx = \text{Area under line.}$$

$$= 1 + 2 + 3 + 4$$

$$= \underline{10}$$

$y = [e^x] \qquad 0 \le x < \ln n$



$$\int_0^{\ln n} [e^x] \, dx = \text{Area under lines.}$$

$$
\begin{aligned}
\text{Area} = \ & (n-1)\{\ln n - \ln(n-1)\} \\
+ \ & (n-2)\{\ln(n-1) - \ln(n-2)\} \\
+ \ & (n-3)\{\ln(n-2) - \ln(n-3)\} \\
+ \ & (n-4)\{\ln(n-3) - \ln(n-4)\} \\
& \ \vdots \\
+ \ & 4\{\ln 5 - \ln 4\} \\
+ \ & 3\{\ln 4 - \ln 3\} \\
+ \ & 2\{\ln 3 - \cancel{\ln 2}\} \\
+ \ & 1\{\cancel{\ln 2} - \ln 1\}
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
& \text{Cancel} \\
& (n-1)\ln n \\
& - \ln(n-1) \\
& - \ln(n-2) \\
& - \ln(n-3) \\
& - \ln(n-4) \\
& \ \vdots \\
& - \ln 4 \\
& - \ln 3 \\
& - \ln 2 \\
& - \ln 1
\end{aligned}
\quad +
$$

PTO

(3) (ctd)

$$
\begin{aligned}
\text{Area} &= (n-1)\ln n - \{\ln(n-1) + \ln(n-2) + \ln(n-3) + \dots + \ln 2 + \ln 1\} \\
&= (n-1)\ln n - \ln((n-1)!) \\
&= (n-1)\ln n - \ln((n-1)!) + \ln n - \ln n \\
&= \underline{n \ln n - \ln(n!)}
\end{aligned}
$$