Attempt the questions yourself before looking at the hints or solutions!

Question:

3 The n positive numbers $x_1, x_2, \dots, x_n$, where $n \geqslant 3$, satisfy

$$x_1 = 1 + \frac{1}{x_2}, \quad x_2 = 1 + \frac{1}{x_3}, \quad \dots, \quad x_{n-1} = 1 + \frac{1}{x_n},$$

and also

$$x_n = 1 + \frac{1}{x_1}.$$

Show that

(i) $x_1, x_2, \dots, x_n > 1$,

(ii) $x_1 - x_2 = -\dfrac{x_2 - x_3}{x_2 x_3}$,

(iii) $x_1 = x_2 = \cdots = x_n$.

Hence find the value of x_1.

Worked Solution: (Want a video solution? Click here)

3. $x_1 = 1 + \frac{1}{x_2}, \ldots\ldots, x_{n-1} = 1 + \frac{1}{x_n}, \quad x_n = 1 + \frac{1}{x_1}$

(i) Since $x_1, \ldots x_n > 0 \Rightarrow \frac{1}{x_1}, \ldots, \frac{1}{x_n} > 0$

$\Rightarrow 1 + \frac{1}{x_1}, \ldots, 1 + \frac{1}{x_n} > 1$

(ii) $x_1 - x_2 = 1 + \frac{1}{x_2} - \left(1 + \frac{1}{x_3}\right)$

$= 1 - 1 + \frac{1}{x_2} - \frac{1}{x_3}$

$= \frac{x_3 - x_2}{x_2 x_3} = \frac{-\,x_2 - x_3}{x_2 x_3}$

(iii) Generally $x_{k-1} - x_k = -\frac{x_k - x_{k+1}}{x_k x_{k+1}}$

It isn't possible for $x_{k-1} = x_k$ and $x_k \neq x_{k+1}$ or vice versa.

Is it possible for all $\{x_i\}$ to be different? No! This is because:

$x_1 - x_2 = -\frac{x_2 - x_3}{x_2 x_3}$

$= \frac{x_3 - x_4}{x_2 x_3^2 x_4}$ and so on, until $x_1 - x_2$ can be equated with an expression with a numerator of $\pm(x_1 - x_2)$ and a positive denominator. This can only happen if $x_1 = x_2$

$\Rightarrow x_1 = x_2 = \ldots = x_n$.

$x_1 = 1 + \frac{1}{x_1} \Rightarrow x_1^2 - x_1 - 1 = 0$

$\Rightarrow x_1 = \frac{1 + \sqrt{5}}{2}$. take positive root since $x_1 > 0$.

(This is the (positive) value of the 'Golden Ratio', the limit of the ratio of successive terms in Fibonacci's Sequence etc.)