Week 7: Sketching Algebraic Functions

STEP 3, 2004

2 The equation of a curve is $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x - 4 - \frac{16(2x+1)^2}{x^2(x-4)}\,.$$

(i) Write down the equations of the vertical and oblique asymptotes to the curve and show that the oblique asymptote is a tangent to the curve.

(ii) Show that the equation $\mathrm{f}(x) = 0$ has a double root.

(iii) Sketch the curve.

STEP 3, 2006

1 Sketch the curve with cartesian equation

$$y = \frac{2x(x^2-5)}{x^2-4}$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence determine the number of real roots of the following equations:

(i) $3x(x^2-5) = (x^2-4)(x+3)\,;$

(ii) $4x(x^2-5) = (x^2-4)(5x-2)\,;$

(iii) $4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)\,.$

Week 8: Vectors in 3 Dimensions

STEP 2, 2000

7 The line l has vector equation $\mathbf{r} = \lambda\mathbf{s}$, where

$$\mathbf{s} = (\cos\theta + \sqrt{3}\,)\,\mathbf{i} + (\sqrt{2}\,\sin\theta)\,\mathbf{j} + (\cos\theta - \sqrt{3}\,)\,\mathbf{k}$$

and λ is a scalar parameter. Find an expression for the angle between l and the line $\mathbf{r} = \mu(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$. Show that there is a line m through the origin such that, whatever the value of θ, the acute angle between l and m is $\pi/6$.

A plane has equation $x - z = 4\sqrt{3}$. The line l meets this plane at P. Show that, as θ varies, P describes a circle, with its centre on m. Find the radius of this circle.

STEP 2, 2010

5 The points A and B have position vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $5\mathbf{i} - \mathbf{j} - \mathbf{k}$, respectively, relative to the origin O. Find $\cos 2\alpha$, where 2α is the angle $\angle AOB$.

(i) The line L_1 has equation $\mathbf{r} = \lambda(m\mathbf{i} + n\mathbf{j} + p\mathbf{k})$. Given that L_1 is inclined equally to OA and to OB, determine a relationship between m, n and p. Find also values of m, n and p for which L_1 is the angle bisector of $\angle AOB$.

(ii) The line L_2 has equation $\mathbf{r} = \mu(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$. Given that L_2 is inclined at an angle α to OA, where $2\alpha = \angle AOB$, determine a relationship between u, v and w.

Hence describe the surface with Cartesian equation $x^2 + y^2 + z^2 = 2(yz + zx + xy)$.

STEP 3, 1998

8 (i) Show that the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{m}$, where $\mathbf{m}$ is a unit vector, intersects the sphere $\mathbf{r}.\mathbf{r} = a^2$ at two points if

$$a^2 > \mathbf{b}.\mathbf{b} - (\mathbf{b}.\mathbf{m})^2 .$$

Write down the corresponding condition for there to be precisely one point of intersection. If this point has position vector $\mathbf{p}$, show that $\mathbf{m}.\mathbf{p} = 0$.

(ii) Now consider a second sphere of radius a and a plane perpendicular to a unit vector $\mathbf{n}$. The centre of the sphere has position vector $\mathbf{d}$ and the minimum distance from the origin to the plane is l. What is the condition for the plane to be tangential to this second sphere?

(iii) Show that the first and second spheres intersect at right angles (*i.e.* the two radii to each point of intersection are perpendicular) if

$$\mathbf{d}.\mathbf{d} = 2a^2 .$$